EXPERIMENT 5: DC AND AC BRIDGE CIRCUITS 10/2/13

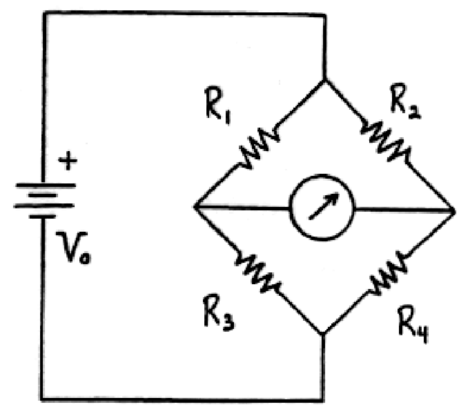
This experiment demonstrates the use of the Wheatstone Bridge for precise resistance measurements and the use of error propagation to determine the uncertainty of a measurement. The bridge circuit will be used to measure an unknown resistance to an accuracy of about 0.1%. We will also construct an AC bridge, and use it to determine the inductance of an unknown inductor.

Before coming to the lab, read the information on error propagation in the appendix to this experiment. You can also do step 1 in both the Wheatstone Bridge and AC Bridge sections.

THE WHEATSTONE BRIDGE

The Wheatstone Bridge circuit can be used to measure an unknown resistance in terms of three known resistances by adjusting one or more of the known resistors to obtain a zero signal (i.e. a "null" reading) on a meter. Such a measurement permits high precision since a very sensitive meter can be used to determine the null condition. The null method also reduces or eliminates sensitivity to a variety of effects (for example, fluctuations in the power supply voltage) which could lead to errors in more conventional measurements.

1. Derive the balance condition for the **D.C. Wheatstone Bridge.**
2. Construct the bridge circuit using the following components:



   $R_1 = \pm$ 0.05% accuracy General Radio resistance box (1Ω steps).

   $R_2 = \pm$ 0.5% accuracy Eico resistance box.

   $R_3 = \pm$ 0.05% accuracy General Radio resistance box (0.1Ω steps).

   $R_4 = \pm$ 0.5% accuracy Eico box (R_E) in parallel with a ± 10% Heathkit resistance box (R_H).

   V_0 = Lambda power supply, set to about 4 V. (You can try larger values — up to 40 volts. This will increase your sensitivity for balancing the bridge, but will introduce more heating which will make the resistance values change. So it's a trade-off. If you want, you can experiment with different values of V_0 when you do part 4.)

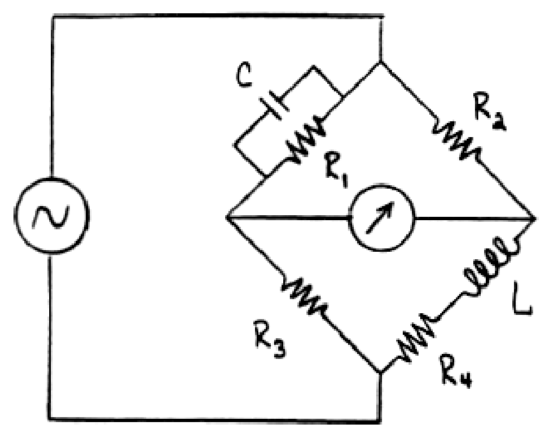
   For the null detector use a DMM (as a voltmeter). Start by setting $R_1 = R_E = 950\ \Omega$, $R_2 = 1000\ \Omega$, $R_3 = 900\ \Omega$ and $R_H = 1\ \text{M}\ \Omega$. Then adjust any or all of the resistors (make only small changes, and keep $R_H > 200\ \text{k}\Omega$) to balance the bridge as accurately as possible.
   Record all the final resistance values and the DMM reading.
3. Now calculate the ratios R_1/R_3 and R_2/R_4 from the resistor settings. In addition, calculate the uncertainty in each of the two ratios, assuming that the accuracies listed in part 2 are correct. To find the uncertainty in R_2/R_4 you will first need to calculate the uncertainty in R_4 (see appendix).
   According to the balance equation R_1/R_3 and R_2/R_4 should be equal. Do your calculated ratios agree to within their calculated uncertainties?

4. Estimate the uncertainty in matching the ratios due to the sensitivity you have in detecting the null condition with the DMM. Start at null and try adjusting the smallest steps in R_3 until you decide you can just definitely tell you have a positive voltage at the null detector, then adjust the other way until you are sure you have the smallest secure negative reading. What is the difference between these? Use this to determine how accurately you have measured the ratio R_2/R_4.

5. Now substitute an "unknown" resistor $R_U = 680\ \Omega$ for R_1 (use a second Heathkit ± 10% resistance box for the unknown). Rebalance the bridge by adjusting R_3. Do not change R_2 or R_4. Calculate R_U from the balance equation, $R_U = R_3 \times (R_2/R_4)$. Then calculate the uncertainty in R_U.

6. Finally, measure R_U directly with a DMM. Compare all your results including the estimated errors in a table. The accuracy of the DMM measurement can be found in Appendix C.

Next we will use the **AC Wheatstone Bridge** circuit shown here to measure an unknown inductor (with L somewhere between 15 and 25 mH).



1. Show that when the bridge is balanced the resistance R_4 and inductance L are given by $R_4 = R_2 R_3 / R_1$ and $L = R_2 R_3 C$. Although these results are independent of ω the determination of L is not very precise for low frequencies because the voltage across L is too small. We will use a value of ω that makes the magnitudes of the impedances of R_4 and L comparable.

2. Construct the bridge with $R_1 \approx R_2 \approx 2000\ \Omega$ and $R_3 \approx R_4 \approx 150\ \Omega$. Use the high precision (General Radio) resistors for R_2 and R_3 and the Eico resistors for R_1 and R_4. Use the Eico capacitance box and the numbered inductor board. Use the function generator with $f \approx 1$ kHz and the output amplitude set to its maximum value as the voltage source. Set the DMM null meter to read AC voltage. Now adjust C and R_1 to minimize the DMM reading. (Because of noise pickup you may only be able to null the bridge to a few mV. You can test whether you have nulled the f = 1 kHz signal by turning the function generator amplitude to zero and observing what happens to the null reading.) Record the results and calculate L. Also, record the number on the inductor board.

3. Change f by a factor of 2 and rebalance the bridge to verify that the balance equations do not depend on ω.

4. The last step is to estimate the uncertainty in L. With f back at 1 kHz, vary C from the null setting as you did in part 4 of the DC bridge and make an estimate of how accurately C can be set. Now estimate the uncertainty in L taking into account the accuracy of the capacitance box itself (± 1%) as well as the precision with which C can be set (the uncertainties in R_2 and R_3 are negligible).

5. Check with your lab instructor to get the actual value of L for the board you used. How close was your measurement to the actual value?

APPENDIX A

ACCURACY, PRECISION, ERRORS, UNCERTAINTY, ETC.

Part of making and reporting a measurement is deciding how accurate it might be. Finding that a distance is 10.0000 cm ± .0001 cm can tell you something very different from 10 cm ± 1 cm. In common speech, the words accuracy and precision are often used interchangeably. However, many scientists like to make a distinction between the meanings of the two words. Accuracy refers to the relationship between a measured quantity and the real value of that quantity. The accuracy of a single measurement can be defined as the difference between the measured value and the true value of the quantity. Since in most cases you don't know the true value (if you did, you wouldn't be bothering to measure it!), you seldom know the true accuracy of your answer. Exceptions to this occur primarily when you are testing an apparatus or new measurement method, and in teaching labs like this one. Since here we often do know the true value, or have measured the same quantity two different ways, whenever you have this opportunity you should always compare the achieved ***accuracy*** (the difference between the true value and your measured value), or the ***consistency*** (the difference between your two determinations) with your independently estimated error as described below.

The words error and uncertainty are often used interchangeably. Nevertheless, it is important to be aware of the distinction between the actual error in a given measurement (i.e. in the amount by which the measured value differs from the true value) and the **uncertainty** in a measurement, which is your estimate of the **most likely magnitude** of this error. The point is that in most experiments we do not know the true value of the quantity we are measuring, and therefore cannot determine the actual error in our result. However, it is still possible to make an estimate of the uncertainty (or the "probable error") in the measurement based on what we know about the properties of the measuring, instruments, etc.

The word precision refers to the amount scatter in a series of measurements of the same quantity, assuming they have been recorded to enough significant figures to show the scatter (you should try to record just enough figures to show this scatter in the last significant figure, or possibly the last two). It is possible for a measurement to be very precise, but at the same time not very accurate. For example, if you measure a voltage using a digital voltmeter that is incorrectly calibrated the answer will be precise (repeated measurements will give essentially the same result to several decimal places) but inaccurate (all of the measurements will be wrong).

Measurement uncertainties can be divided into two distinct classes: ***random*** or ***statistical*** errors, and ***systematic*** errors. Systematic errors are things like the voltmeter calibration error mentioned above, or perhaps that you made all your length measurements with a metal tape measure that had expanded because you were in a much warmer room than the one where the tape was constructed. Systematic errors can be quite difficult to estimate, since you have to understand everything about how your measurement system works.

Somewhat counter-intuitively, the random error is usually easier to estimate. It is due to some combination of the limited precision to which a quantity can be read from a ruler or meter scale, and intrinsic "noise" on the measurement. For example, if a radioactive source that gives an average of one count per second is counted for exactly 100 seconds, you will find that you don't always get exactly 100 counts even if your count perfectly accurately (no ***mistakes***). About one-third of the time, you will get fewer than 90 or more than 110 counts, and occasionally (about 0.5% of the time) you will get fewer than 70 or more than 130 counts. If you make a plot of the distribution of a large number of 100-second counts, you will get a curve called a "Poisson distribution." Unless the average number of counts is small, this curve will be very close to a gaussian or "bell curve". Most random errors follow this kind of distribution. The expected size of the uncertainty in a measurement is described by the width of this curve. The limits that contain 2/3 of the measurements (±10 in our example) are called the "1-sigma" uncertainty. If the errors follow the bell curve, then 95% of the results will be within ±2σ, and 99.5% within ±3σ.

You can often estimate the random error in a measurement empirically. If you can make several *independent* measurements of some quantity, you can obtain an estimate of the precision of each individual measurement. (The "independent" part is important: if you measure a length with a meter stick, and on the first try estimate 113.3 mm, you are likely to write down 113.3 on subsequent measurements as well, even if you can really only estimate to ±0.1 or 0.2 mm. One way around this is to have different people make each measurement, and write them down without looking at each other's answers. Or by yourself, you could start from a random point on the ruler each time and estimate the readings at both ends, then do all the subtractions afterwards.)

The following example illustrates several of these ideas. In this example the resistance of a known 1000 ± 0.01 Ω resistor is determined by measuring V and I for several different voltage settings. The results are given in the table on the following page. The average value of R in this example is 1002.4 Ω, so our final result has an error of 2.4 Ω. The precision of any individual measurement of R can be determined by calculating the standard deviation:

$$\sigma \equiv \frac{1}{\sqrt{N-1}}\left[\sum_{i=1}^{N}\left(x_i-\bar{x}\right)^2\right]^{1/2}$$

where $\bar{x}$ is the average value of x and where N is the number of measurements. In this example the standard deviation is 5.6 Ω. We could take this as an estimate of the uncertainty or probable error, since any individual measurement has a reasonable probability of being in error by at least that amount. It should be emphasized, however, that the <u>actual</u> error in a measurement can be much larger than the standard deviation if there are systematic errors (for example errors in the calibration of some meter) that affect all the measurements the same way.

Data for a 1000 ± 0.01 kΩ Resistor

V(a) (volts)	I(b) (mA)	R(c) (Ω)
1.000	0.99	1010
2.000	1.99	1005
3.000	3.00	1000
4.000	4.02	995
5.000	4.99	1002

Average = 1002.4
Standard Deviation = 5.6
Error = 2.4
% Error = 0.24 %

(a) Measured with digital voltmeter.
(b) Measured with Simpson VOM.
(c) Calculated from R = V/I.

Propagation of Errors

In many experiments, our desired result Q is determined from a mathematical formula that uses two or more separately measured quantities: $Q = f(x_i, \ldots x_n)$, where $x_1, \ldots\ x_n$ are measured values, and f is the mathematical function. If each of the x_i were to change by an amount δx_i, then to first order Q will change by

$$\Delta Q = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \Delta x_i . \qquad (1)$$

We have estimated the uncertainties in the n measured quantities, and want to calculate the uncertainty in Q. We know the expected magnitude of Δx_i, but expect it is equally likely to be positive or negative, so its average value would be zero. We usually try to estimate (or assume we know) the quantity $\sigma_{x_i} = \left\langle \Delta x_i^2 \right\rangle^{1/2}$, or the square root of the average of Δx_i^2 (the "root mean square" or r.m.s. value of the expected error). The problem of figuring out the uncertainty in the result, given the formula and the uncertainties in the numbers going into it, is called "error propagation."

Since $\langle \Delta Q \rangle$, the average or "expected" value of ΔQ is also zero, we need to calculate the expected value of ΔQ^2:

$$\left\langle \Delta Q^2 \right\rangle = \left\langle \left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \Delta x_i \right)^2 \right\rangle . \qquad (2)$$

Taking the square will produce terms of the form $coeff \cdot \left(\Delta x_i \Delta x_j\right)^2$. For $i \neq j$ we generally assume the expected value is zero, since if Δx_i is positive, Δx_j should be equally likely to be positive or negative. This assumes that the errors in Δx_i and Δx_j are ***independent***. If this is not true, you must keep these terms! The expected value of the terms with $i = j$ are $\left\langle \Delta x_i^2 \right\rangle$, which is just $\sigma_{x_i}{}^2$.

There are just two cases that in combination will cover 98% of the error propagation problems you will run into. We give these results here with the recommendation that you memorize them, although all can easily be derived from equation (2) above. A and B are two measured (or calculated) values:

For $Q = A + B$ **or** $Q = A - B$: $\sigma_Q \equiv \left\langle \Delta Q^2 \right\rangle^{1/2} = \sqrt{\sigma_A^2 + \sigma_B^2}$. (Note errors add even for $Q = A - B$.)

For $Q = A \cdot B$ or $Q = A/B$: $\dfrac{\sigma_Q}{Q} = \sqrt{\left(\dfrac{\sigma_A}{A}\right)^2 + \left(\dfrac{\sigma_B}{B}\right)^2}$.

For independent errors, it makes some sense that the errors add as the square root of the sum of their squares, or "in quadrature": the errors might add, or they might have opposite signs, and at least partially cancel. So on the average, we could expect them to add "at right angles". Beyond that, the two formulae above are easily remembered as "for addition or subtraction, add absolute errors, for multiplication or division, add percentage errors." (Where "add" means "add in quadrature".) One other occasionally useful result is for $Q = A^n$, $\dfrac{\sigma_Q}{Q} = n\dfrac{\sigma_A}{A}$.

For a mixed case like $Q = (A + B)/(C + D)$, you first add absolute errors for the numerator and denominator, then convert these to % errors, and add them to get the error in Q. So error propagation becomes largely an exercise in converting back and forth from absolute to percentages.

BRIDGE CIRCUITS

You do have to be careful of correlated errors. This actually happens most often when it's really the same quantity that shows up in more than one place. In that case, the errors are perfectly correlated. Take the case of $Q = 2A$, which could also be written $Q = A + A$. If you use the formulae given above for independent errors, you'll get different answers for σ_Q! If you use equation (2) and keep the cross term, they'll come out the same. A more subtle example comes up in experiment 5: R_4 in the Wheatstone Bridge consists of R_E (0.5% accuracy) in parallel with R_H (10% accuracy), so that:

$$R_4 = \frac{R_E R_H}{R_E + R_H}. \qquad (3)$$

You can calculate the errors in the numerator and denominator separately using the independent error formulae, but then you can't combine them with the independent errors formula because they contain the same variables, so these errors aren't independent.

To do such cases exactly, it's usually easiest (and always safest) to go back to equation (2), which gives:

$$\frac{\sigma_{R_4}}{R_4} = \left(\left(\frac{R_4}{R_E}\right)^2 \left(\frac{\sigma_{R_E}}{R_E}\right)^2 + \left(\frac{R_4}{R_H}\right)^2 \left(\frac{\sigma_{R_H}}{R_H}\right)^2 \right)^{1/2}. \qquad (4)$$

But you can often save a huge amount of effort by looking at the magnitude of the numbers and making approximations. In this case, typically R_E = 1 kΩ and R_H = 200 kΩ. Although the fractional error in R_H is large, its error doesn't contribute much to the total error in R_4 since it is multiplied by the square of a factor ($R_4/R_H \approx 1/200$) that is small compared to 1. You could have told this without bothering to derive equation (4): looking at equation (3), $R_H >> R_E$, so the denominator $\approx R_H$, and this will approximately cancel the R_H in the numerator. So $R_4 \approx R_E$ and has the same uncertainty as R_E, or 0.5%. This you can all do in your head!

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Whenever possible measured values of quantities should be compared with given or theoretical values and the percent error given. This error should be compared with your estimated uncertainty. If the error is less than 1 or 2 times your estimated σ, no comment is required except that your result is "in reasonable agreement" with the accepted value. If you are more than 2σ off, this should happen by chance less than one time in 20, so you should look for mistakes or discuss possible systematic errors that were not included in your estimate.

In some of the experiments you will be asked to make detailed calculations of the uncertainties in your measurements. This is usually not required since the calculations are often long and time consuming to do exactly. But it is always important for experimenters to have an approximate idea of the uncertainties in their results. With suitable approximations, by ignoring variables that make insignificant contributions, and using the two simple independent-error results, you can do most of the error estimation in your head, and put down ± σ. The usual convention is to convert x to the same units as the result and give the absolute error. Generally one significant figure is more than adequate for errors, and in some cases just being sure to round your result to an appropriate number of significant figures is enough and you don't bother to write down the ± σ.

More discussion of errors and detailed derivations can be found in *Data Reduction & Error Analysis for the Physical Sciences*, 3rd Edition by Bevington & Robinson (QA278 B48 2003).